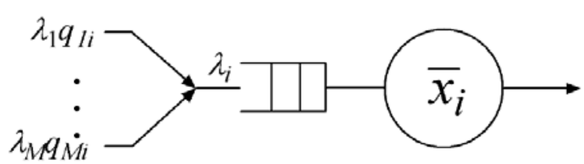
# Closed Network Model

Example Results



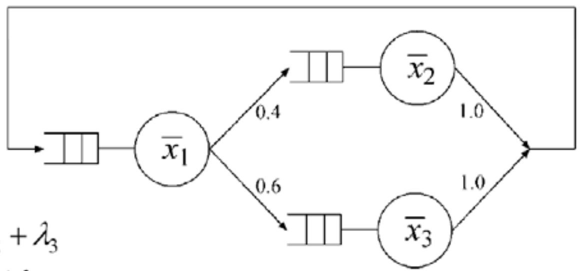
# Total Arrival Rate to Server *i*




$\lambda_i = \sum_{j=1}^{M} \lambda_j q_{ji}$ for $i = 1, 2, \ldots, M$

- can only determine relative values of $\lambda_i$'s

# Relative Arrival Rate




$$\lambda_1 = \lambda_2 + \lambda_3$$
$$\lambda_2 = 0.4\lambda_1$$
$$\lambda_3 = 0.6\lambda_1$$
$$\Rightarrow \frac{\lambda_2}{\lambda_1} = 0.4 \text{ and } \frac{\lambda_3}{\lambda_1} = 0.6$$



# Relative Utilization

- Can determine the relative values of $U_i$'s
- Example

  For $\overline{x}_1 = 4.0, \overline{x}_2 = 2.0, \overline{x}_3 = 5.0$

  we have $\frac{U_2}{U_1} = \frac{\lambda_2 \overline{x}_2}{\lambda_1 \overline{x}_1} = (0.4)\frac{2.0}{4.0} = 0.2$

  $$\frac{U_3}{U_1} = \frac{\lambda_3 \overline{x}_3}{\lambda_1 \overline{x}_1} = (0.6)\frac{5.0}{4.0} = 0.75$$

- Can also determine the upper bound of $U_i$'s
- Example

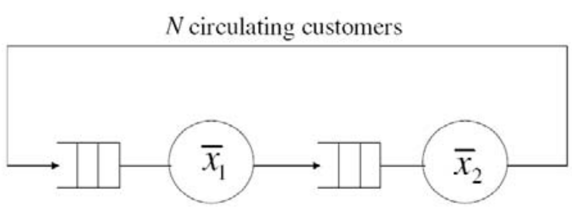
  $U_1 \le 1.0 \Rightarrow U_2 \le 0.2$ and $U_3 \le 0.75$

# Closed Network Model

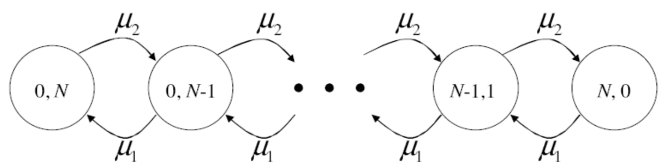
Analytic Results



# Cyclic Queue with Two Servers



# State Transition Diagram





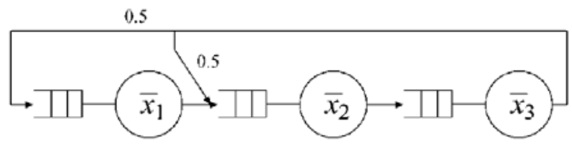

# Product Form Solution

$$P(n_1, n_2) = \frac{1}{G(N)} \prod_{i=1}^{2} y_i^{n_i} \qquad \text{where } y_i = \frac{1}{\mu_i}$$

$$G(N) = \sum_{n_1+n_2=N} \prod_{i=1}^{2} y_i^{n_i}$$

## Product Form Solution

- Let $e_i$ be a solution to
  $\lambda_i = \sum_{j=1}^{M} \lambda_j q_{ji}, \; i = 1, 2, \ldots, N$
- Example




- Solution
  $e_1 = c$ (any constant), $e_2 = 2c$, $e_3 = 2c$



## Product Form Solution

- Let $y_i = \dfrac{e_i}{\mu_i}$
  $P(n_1, n_2, \ldots, n_M) = \dfrac{1}{G(N)} \prod_{i=1}^{M} y_i^{n_i}$
  where $G(N)$ is a normalization constant and $N$ is total no. of customers in network

## Normalization Constant

- $\sum_{\text{all feasible states}} \mathrm{P}(n_1, n_2, \ldots, n_M) = 1$
- A feasible state is characterized by:
  $n_1 + n_2 + \ldots + n_M = N$ and $n_i \geq 0$
- $G(N) = \sum_{n_1+n_2+\ldots+n_M=N} \prod_{i=1}^{M} y_i^{n_i}$



## Normalization Constant

- *Note*:
  - the value of steady state probabilities is not affected by the value of $c$ used in the solution $e_i$
  - no. of feasible states $= \binom{N+M-1}{N}$
  - example: for $N = 3$, $M = 3$

    feasible states: (3,0,0),(2,1,0),(2,0,1),(1,2,0),(1,0,2),(1,1,1)
    (0,2,1),(0,1,2),(0,3,0),(0,0,3)

    total no. $= \binom{5}{3} = 10$

# Performance Measures

- Mean arrival rate to server $i$ : $\lambda_i = \frac{G(N-1)}{G(N)} e_i$
- Utilization factor at server $i$ : $U_i = \frac{\lambda_i}{\mu_i} = \frac{G(N-1)}{G(N)} y_i$
- Mean no. of customers at server $i$ : $\overline{n}_i = \sum_{k=0}^{N-1} \frac{G(N-k-1)}{G(N)} y_i^{k+1}$
- Mean delay at server $i$ : $\overline{t}_i = \frac{\overline{n}_i}{\lambda_i}$



# Observation

- Can express performance measures as a function of $G(N)$
- Computing $G(N)$ by a direct enumeration of all feasible states is not efficient
  - Example: N = 10, M = 10

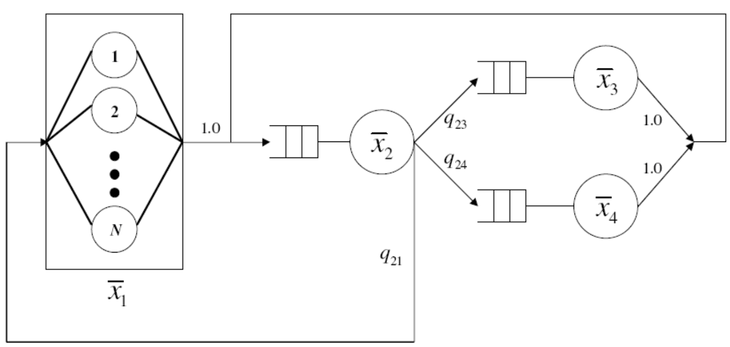
    Number of feasible states = $\binom{19}{10} \approx 92400$
- Desirable to have an efficient procedure to compute $G(N)$

# Closed Network Model –Infinite Server Case

- Analysis so far is based on the case of a single server at each service facility
- Consider the special case where one of the service facilities has infinite servers
- Interactive system model



# Interactive System Model – Fundamental Results



## Relative Arrival Rates

- Let $e_i$ be a solution to

  $\lambda_i = \sum_{j=1}^{M} \lambda_j q_{ji} \qquad i = 1, 2, \ldots, M$
- Example on previous page

  $\lambda_1 = q_{21}\lambda_2 \qquad \lambda_2 = \lambda_1 + \lambda_3 + \lambda_4$

  $\lambda_3 = q_{23}\lambda_2 \qquad \lambda_4 = q_{24}\lambda_2$
- Solution

  $e_1 = q_{21}, \quad e_2 = 1, \quad e_3 = q_{23}, \quad e_4 = q_{24}$



## Server Utilization

- Let $y_i = e_i \overline{x}_i \qquad i > 0$
- Relative utilization: $\dfrac{U_i}{U_j} = \dfrac{e_i \overline{x}_i}{e_j \overline{x}_j} = \dfrac{y_i}{y_j} \quad i, j > 0$
- Let $U_i(N)$ = utilization of server $i$ when no. of customers is $N$ (or mean number of customers in thinking state when $i = 1$)

  At $N = 1$,

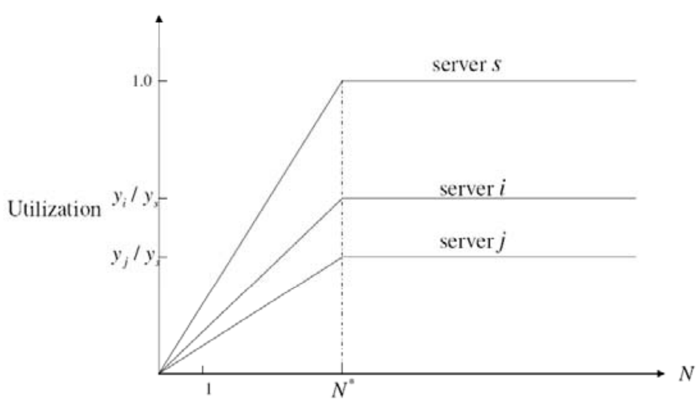
  $$\sum_{j=1}^{M} U_j(1) = 1 \;\Rightarrow\; U_i(1) = \frac{y_i}{\sum_{j=1}^{M} y_j}$$

# Upper Bound Utilization

- Let $s = max_{i>1} \{y_i\}$
  i.e., server $s$ has highest utilization
- $N^* = \dfrac{1}{U_s(1)}$
- Upper bound:
  For $1 < N \leq N^*$, $U_i(N) \leq NU_i(1)$
  For $N \geq N^*$, $U_i(N) \leq \dfrac{y_i}{y_s}$



# Upper Bound Utilization



# Lower Bound Mean Response Time

- Let $b_i = \dfrac{\lambda_i}{\lambda_1}, \quad \overline{T}(1) = \sum_{i>1} b_i \overline{x}_i$
- Lower bound: from previous results, $\overline{T}(N) = \dfrac{N}{\lambda_1} - \overline{x}_1$

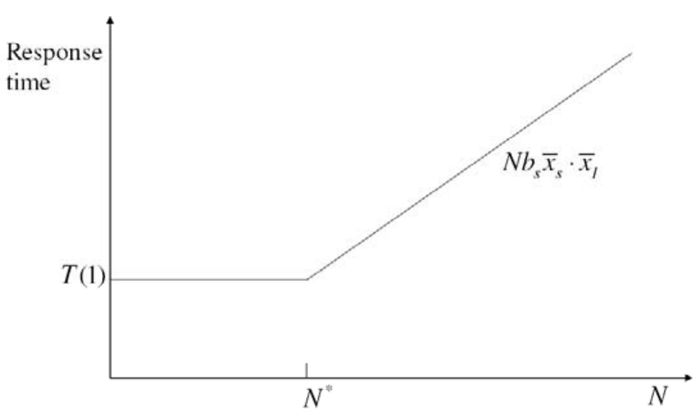
$U_s(N) = \lambda_s \overline{x}_s < 1$ implies that $\lambda_s < \dfrac{1}{\overline{x}_s}$

$b_s = \dfrac{\lambda_s}{\lambda_1} < \dfrac{1}{\lambda_1 \overline{x}_s}$ or $\lambda_1 < \dfrac{1}{b_s \overline{x}_s}$

We thus have: $\overline{T} > N b_s \overline{x}_s - \overline{x}_1$



# Lower Bound Mean Response Time



# Heavy Load Approximation

- At heavy load, i.e., $N >> N^*$, can use the following approximations
  - server utilization: approximated by upper bound

    $$U_s \approx 1.0 \quad U_i \approx \frac{y_i}{y_s}$$

  - mean response time: approximated by lower bound

    $$\overline{T}(N) \approx N b_s \overline{x}_s - \overline{x}_1$$



# Interactive System Model –Analytic Results

- Product form solution still holds

$$P(n_1, n_2, ..., n_M) = \frac{1}{G(N)} F_1(n_1) F_2(n_2) ... F_M(n_M)$$

where $F_1(n_1) = \frac{y_1^{n_1}}{n_1!}$, and $F_i(n_i) = y_i^{n_i}$, $i = 2, ..., M$

and $G(N) = \sum_{\text{all feasible states}} F_1(n_1) F_2(n_2) ... F_M(n_M)$

# Interactive System Model –Analytic Results

- Utilization factor of server $i$ (for $i = 2, 3, \ldots, M$)

$$U_i = \frac{G(N-1)}{G(N)} y_i$$

- Mean number at server $i$ (for $i = 2, 3, \ldots, M$)

$$\overline{n}_i = \sum_{k=0}^{N-1} \frac{G(N-k-1)}{G(N)} y^{k+1}$$

- Mean response time (seen by users at terminals)

$$\overline{T} = \frac{N}{\lambda_1} - \frac{1}{\mu_1}$$

where $\lambda_1 = \frac{G(N-1)}{G(N)} e_1$