Open Network Models

Example Results

Notation

M = number of servers (nodes)

γ_i = external arrival rate to server i

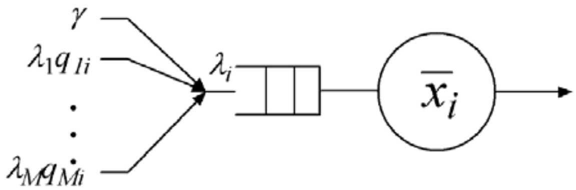
λ_i = total arrival rate (external and internal) to server i

$\overline{X_i}$ = mean service time at server i

q_{ij} = transition probability (from server i to server j)

U_i = utilization factor of server i

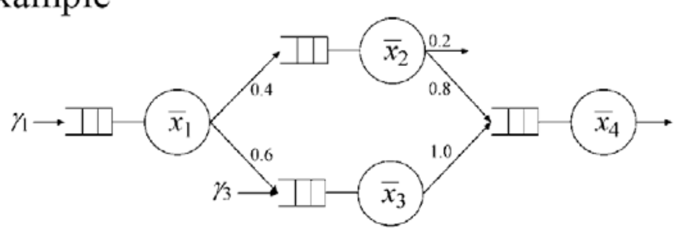
Total Arrival Rate to Server *i*



$$\lambda_i = \gamma_i + \sum_{j=1}^{M} \lambda_j q_{ji} \quad \text{for } i = 1, 2, \ldots, M$$

Total Arrival Rate to Server *i*

- Can determine the λ_i's uniquely
- Example



Let $\gamma_1 = 2.0$ and $\gamma_3 = 3.0$

$\lambda_1 = \gamma_1 = 2.0$

$\lambda_2 = 0.4\lambda_1 = 0.8$

$\lambda_3 = \gamma_3 + 0.6\lambda_1 = 4.2$

$\lambda_4 = 0.8\lambda_2 + \lambda_3 = 4.84$

Utilization of Server *i*

- Can determine the utilization factor, U_i's, uniquely

  $U_i = \lambda_i \overline{x}_i$

- Example

  for $\overline{x}_1 = 0.25$, $\overline{x}_2 = 0.5$, $\overline{x}_3 = 0.15$, $\overline{x}_4 = 0.125$,

  we have $U_1 = 0.5$, $U_2 = 0.4$, $U_3 = 0.63$, $U_4 = 0.605$

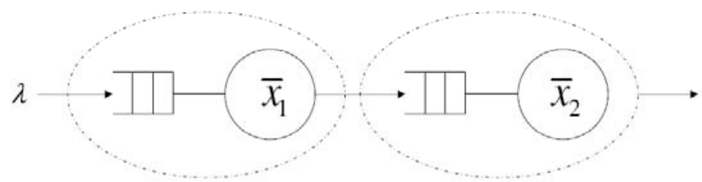
Open Network Models

Analytic Results

Assumptions

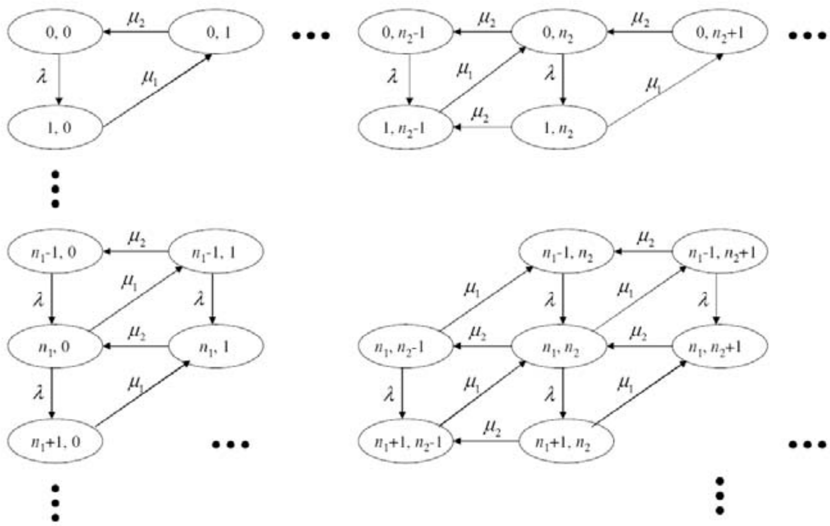
- External arrivals -interarrival time is exponentially distributed
- Single server at each service facility
- Service time is exponentially distributed
- Infinite queues

Example –Two-Stage Tandem Queue



- Exponential interarrival and service times
  - $-\lambda$ = arrival rate
  - $-\overline{x}_i = \dfrac{1}{\mu_i}$ = mean service time at stage i, $i = 1, 2$
- State of model is (n_1, n_2) where n_1 and n_2 are numbers of customers in first and second stage respectively

State Transition Diagram



Balance Equations

$$\lambda P(0,0) = \mu_2 P(0,1)$$

$$(\lambda + \mu_2) P(0, n_2) = \mu_1 P(1, n_2 - 1) + \mu_2 P(0, n_2 + 1), \qquad n_2 > 0$$

$$(\lambda + \mu_1) P(n_1, 0) = \lambda P(n_1 - 1, 0) + \mu_2 P(n_1, 1), \qquad n_1 > 0$$

$$(\lambda + \mu_1 + \mu_2) P(n_1, n_2) = \lambda P(n_1 - 1, n_2) + \mu_1 P(n_1 + 1, n_2 - 1) + \mu_2 P(n_1, n_2 + 1), \; n_1, n_2 > 0$$

Solution Method

- Define normalization equation

  $\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P(n_1, n_2) = 1$
- To solve for $P(n_1, n_2)$, one technique is to assume a solution form and see if the solution satisfies the balance equations
- Assume a product form solution

  $$P(n_1, n_2) = \frac{1}{G} \rho_1^{n_1} \rho_2^{n_2}$$

  where $\rho_i = \dfrac{\lambda}{\mu_i}, i = 1, 2$ and G = normalization constant

Verification

i. Balance equation for state (0,0)

$$\text{LHS (flow out)} = \frac{1}{G}\lambda$$

$$\text{RHS (flow in)} = \mu_2 \frac{1}{G} \rho_2 = \frac{\lambda}{G} = \text{LHS}$$

Verification

ii. Balance equation for state $(0, n_2)$

$$\text{LHS} = (\lambda + \mu_2)\frac{1}{G}\rho^{n_2}$$

$$\text{RHS} = \mu_1 \frac{1}{G}\rho_1\rho_2^{n_2-1} + \mu_2 \frac{1}{G}\rho_2^{n_2+1} = \frac{1}{G}\mu_2\rho_2^{n_2} + \frac{1}{G}\lambda\rho_2^{n_2} = \text{LHS}$$

- similar verification can be done for states $(n_1, 0)$ and (n_1, n_2), $n_1, n_2 > 0$

Product Form Solution

- $P(n_1, n_2) = \frac{1}{G}\Pi_{i=1}^{2}\rho_i^{n_i}$
- Evaluation of G

  normalization equation $\Rightarrow \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{1}{G}\rho_1^{n_1}\rho_2^{n_2} = 1$

  or $G = \frac{1}{(1-\rho_1)(1-\rho_2)}$ $\quad \rho_1, \rho_2 < 1$

- Thus, $P(n_1, n_2) = \prod_{i=1}^{2}(1-\rho_i)\rho_i^{n_i}$

Performance Measures

i. Mean number of customers at server i: $\overline{n}_i = \frac{\rho_i}{1-\rho_i}$, $i = 1, 2$

ii. Mean number of customers in network: $\overline{N} = \sum_{i=1}^{2} \frac{\rho_i}{1-\rho_i}$

iii. Mean response time at server i: $\overline{t}_i = \frac{1}{\mu_i(1-\rho_i)}$, $i = 1, 2$

iv. Mean end-to-end delay: $\overline{T} = \frac{\overline{N}}{\lambda} = \frac{1}{\lambda}\sum_{i=1}^{2} \frac{\rho_i}{1-\rho_i} = \overline{t}_1 + \overline{t}_2$

General Model - Notation

M = number of servers

γ_i = external arrival rate to server i

λ_i = total arrival rate (external and interval) to server i

μ_i = mean service rate at server i

q_{ij} = transition probability (from server i to server j)

Thus, Pr[departure from network after visting server i]

= $1 - \sum_{j=1}^{M} q_{ij}$

Product Form Solution

- Let state be $(n_1, n_2, \ldots, n_M)$ where n_i= number of customers at server i
- $P(n_1, n_2, \ldots, n_M) = \frac{1}{G} \prod_{i=1}^{M} \rho_i^{n_i}$
- Evaluation of G
  - G can be evaluated by the normalization equation, i.e.,

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} P(n_1, n_2, \ldots, n_M) = 1$$

$$\Rightarrow \frac{1}{G} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} \prod_{i=1}^{M} \rho_i^{n_i} = 1$$

or $G = \prod_{i=1}^{M} \frac{1}{1-\rho_i}$, if $\rho_i < 1$ for all i

Performance Measures

i. Mean no. at server i : $\bar{n}_i = \frac{\rho_i}{1-\rho_i}$

ii. Mean no. in network : $\bar{N} = \sum_{i=1}^{M} \frac{\rho_i}{1-\rho_i}$

iii. Mean response time at server i : $\bar{t}_i = \frac{1}{\mu_i(1-\rho_i)}$

iv. Mean end-to-end delay : $\bar{T} = \frac{\bar{N}}{\gamma} = \frac{1}{\gamma} \sum_{i=1}^{M} \frac{\rho_i}{1-\rho_i}$ feedback

where $\gamma = \sum_{i=1}^{M} \gamma_i$ is the total external arrival rate

Open Network Model, Infinite Server Case –Fundamental Results

- Analysis so far is based on the case of a single server at each service facility
- Suppose one or more service facilities have infinite servers (i.e., no queueing)
- Fundamental results
  - for service facility *i* with infinite servers:
    $U_i = \lambda_i \overline{x}_i =$ mean no. of busy servers

    Little's Formula

Open Network Model, Infinite Server Case –Analytic Results

- Product form solution still holds
  $P(n_1, n_2, ..., n_M) = P_1(n_1)P_2(n_2)...P_M(n_M)$
- If server facility *i* has a single server,
  $P_i(n_i) = (1-\rho_i)\rho_i^{n_i}$
- If server facility *i* has infinite servers,
  $$P_i(n_i) = \frac{e^{-\rho_i}\rho_i^{n_i}}{n_i!}$$
  - performance results: $\overline{n}_i = \rho_i$ and $\overline{t}_i = \dfrac{1}{\mu_i}$