Analytic Modeling

M/G/1 Model

Notation

L = length of observation period

n = number of customers served in L (assume server is always busy)

x_j = service time of j^{th} customer

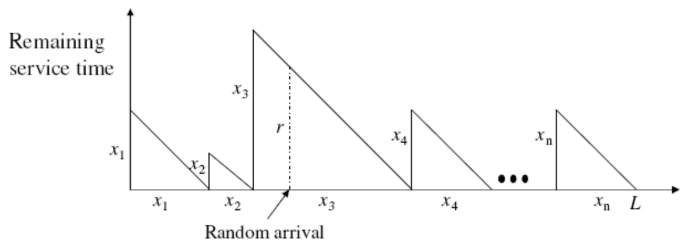
$\overline{x}$ = mean service time $= \frac{1}{n}\sum_{j=1}^{n} x_j$

$\overline{x^2}$ = second moment of service time $= \frac{1}{n}\sum_{j=1}^{n}(x_j)^2$

$\overline{r}$ = mean remaining service time seen by a random arrival

Mean Remaining Service Time (seen by a random arrival)

- Suppose server is always busy



$$\overline{r} = \text{mean height of triangles} = \frac{\text{total area}}{L} = \frac{\frac{1}{2}\sum_{j=1}^{n}(x_j)^2}{\sum_{j=1}^{n} x_j} = \frac{\overline{x^2}}{2\overline{x}}$$

Mean Service Time - FCFS

Number in system, n:	0	1	2	3	…
Mean response time:	$\overline{x}$	$\overline{r}+\overline{x}$	$\overline{r}+2\overline{x}$	$\overline{r}+3\overline{x}$	…

Overall mean response time $\overline{T}$

$$= \Pr[\text{server idle}]\overline{x} + \Pr[\text{server busy}]\overline{r}$$
$$+ \{\Pr[n=1]\overline{x} + \Pr[n=2]2\overline{x} + \Pr[n=3]3\overline{x} + \ldots\}$$
$$= (1-\rho)\overline{x} + \rho\overline{r} + \overline{N}\overline{x}$$

Where $\rho = \lambda\overline{x}$ and $\overline{N}$ = mean no.of customers in system

Mean Response Time

$$\overline{T} = (1-\rho)\overline{x} + \rho\overline{r} + \lambda\overline{T}\overline{x}$$

or

$$\overline{T} = \overline{x} + \frac{\lambda\overline{x^2}}{2(1-\rho)}$$

For the case of exponential service time distribution,

$$\overline{x^2} = 2\overline{x}^2 \text{ and } \overline{T} = \frac{\overline{x}}{1-\rho}$$

Mean Response Time – Priority Scheduling

- Consider a Head-Of-Line (HOL) priority
- Non-preemptive discipline
- Different classes may have different arrival rates and service requirements

Notation

K = number or priority classes

For class k,

λ_k = mean arrival rate

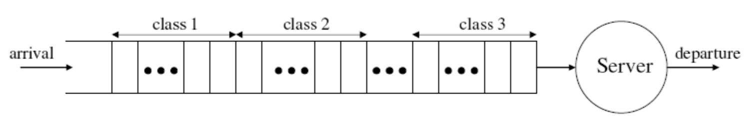
$\overline{x}_k$ = mean service time

$\overline{x^2}_k$ = second moment of service time

$\overline{W}_k$ = mean waiting time

$\rho_k = \lambda_k \overline{x}_k$

Head Of Line Discipline



Derivation of $\overline{W}_k$

Based on a random arrival (say class k)

Referred to as the "Tagged" customer

Define

$\overline{N}_{ik}$ = mean number of class i already in queue when the tagged customer arrives

$\overline{W}_0$ = mean remaining service time of customer in service

$\overline{M}_{ik}$ = mean number of class i arrivals when the tagged customer is in queue

$$\overline{W}_k = \sum_{i=k}^{K} \overline{N}_{ik}\,\overline{x}_i + \sum_{i=k+1}^{K} \overline{M}_{ik}\,\overline{x}_i + \overline{W}_0$$

Derivation of $\overline{W}_k$

- Little's formula, class i, queue only

  $\overline{N}_{ik} = \lambda_i \overline{W}_i$

- Class i arrival

  $\overline{M}_{ik} = \lambda_i \overline{W}_k$

Derivation of $\overline{W}_k$

$\overline{W}_k = \sum_{i=k}^{K} \lambda_i \overline{W}_i \overline{x}_i + \sum_{i=k+1}^{K} \lambda_i \overline{W}_k \overline{x}_i + \overline{W}_0$ or

$$\overline{W}_k = \frac{\sum_{i=k}^{K} \rho_i \overline{W}_i + \overline{W}_0}{1 - \sum_{i=k+1}^{K} \rho_i}$$

where $\overline{W}_0 = \sum_{i=k}^{K} \rho_i \frac{\overline{x^2}_i}{2\overline{x}_i}$ or $\overline{W}_0 = \sum_{i=k}^{K} \frac{\lambda_i \overline{x^2}_i}{2}$

and $\rho_{\mathrm{i}} = \Pr[\text{tagged customer sees class } i \text{ in service}]$

- Can compute $\overline{W}_k$ recursively in the following order $\overline{W}_K,\ \overline{W}_{K-1}, \ldots, \overline{W}_1$