Analytic Modeling

Markovian Queuing Model

Markovian Queuing Model at Steady Rate

- Has memoryless property (next state completely determined by current state)
- Consequence of the use of exponential distributions
- Solution method
  i. Obtain the balance equations
  ii. Solve the balance equations with normalization equation given by
  $\sum_{\text{all states, } s} P(s) = 1$
  and obtain the steady state probability
- Calculate performance measures from steady state probabilities

M/E$_2$/1/K Example

- Erlang distribution - with parameters μ and k (k is an integer)

$$f(x) = \frac{\mu^k x^{k-1} e^{-\mu x}}{(k-1)!}$$

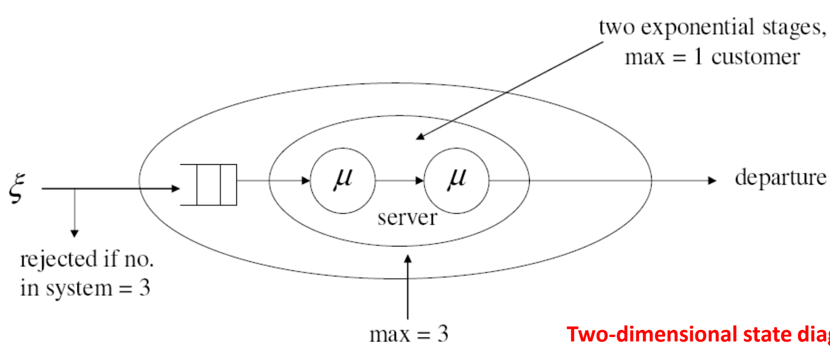
$$F(x) = 1 - \sum_{j=0}^{k-1} \frac{(\mu x)^j e^{-\mu x}}{j!}$$

$$\bar{x} = \frac{k}{\mu}, \quad \sigma^2 = \frac{k}{\mu^2}$$

  - can be interpreted as the sum of k independent random variables, each of them having exponential distribution with parameter μ
  - coefficient of variation < 1 when $k > 1$

M/E$_2$/1/K model for K = 3

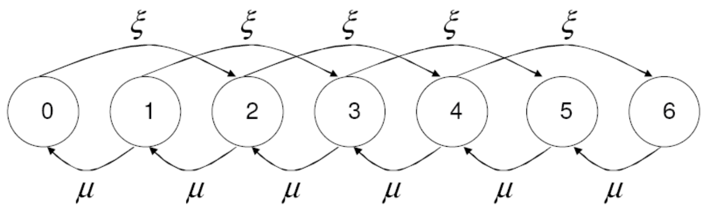
- Can be viewed as



Two-dimensional state diagram

State Transition Diagram

- Can define model states as no. of stages of service in the system



Balance Equations

$$\xi P(0) = \mu P(1)$$
$$(\xi + \mu)P(n) = \xi P(n-2) + \mu P(n+1), \quad n = 2,3,4$$
$$\mu P(5) = \xi P(3) + \mu P(6)$$
$$\mu P(6) = \xi P(4)$$

Normalization equation: $\sum_{n=0}^{6} P(n) = 1$

Performance Measures

- Let $q(n) = \Pr[\text{no. in system } = n]$

  $q(0) = P(0)$

  $q(n) = P(2n-1) + P(2n), \quad n = 1, 2, 3$

- Throughput $= \lambda = \xi[1 - q(3)] \quad (\lambda < \xi)$

- Mean response time

  $$\overline{T} = \frac{\overline{N}}{\lambda} \quad \text{where } \overline{N} = \sum_{n=0}^{3} nq(n)$$

Remarks

- Approach is applicable when no. of states is finite, but may have problems if no. of states is large
- For models with a countable no. of states, must rely on the form of the balance equations and hope to obtain a closed form solution
- Examples
  - i. M/M/1

    $\sum_{n=0}^{\infty} P(n) = 1 \quad \Rightarrow P(0) = 1 - \rho \text{ if } \rho < 1$

    – closed form solution is possible
  - ii. M/E_2/1

    – closed form solution is not possible

Approximation Method

- Restrict analysis to a subset of states
- Example
  - suppose $P(n)$ decreases with n
  - can assume $P(n) = 0$, for $n > n^*$, and solve for $P(n)$ directly
  - n^* should be selected such that no. of states in the subset is not excessive, and the error introduced is acceptable