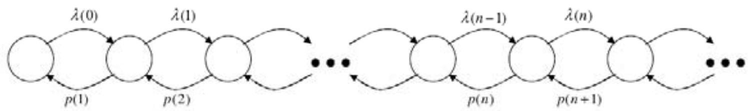
Analytic Modeling

Birth-Death Model Solution Method

Solution Steps

1. Obtain the balance equations
2. Solve the balance equations. This involves the evaluation of G and $P(n)$
3. Compute $\overline{N}$ = mean no. in system = $\sum_{n=0}^{\infty} nP(n)$
4. Compute λ = mean arrival rate = $\sum_{n=0}^{\infty} \lambda(n)P(n)$
5. Use Little's formula to obtain $\overline{T} = \dfrac{\overline{N}}{\lambda}$

State Transition Diagram



- Convenient method to obtain the balance equations by drawing closed curves around each state and equating *rate of flow out to rate of flow in*

Balance Equation

For state $n > 0$

rate of flow out $= [\lambda(n) + \mu(n)]P(n)$

rate of flow in $= \lambda(n-1)P(n-1) + \mu(n+1)P(n+1)$

For state 0

rate of flow out $= \lambda(0)P(0)$

rate of flow in $= \mu(1)P(1)$

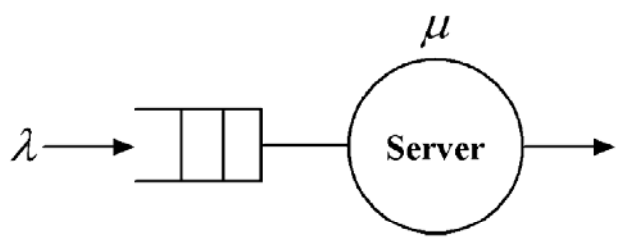
Notation (Kendall)

- A/S/m/B/K
  - A : interarrival time distribution
  - S : service time distribution
  - m : the number of servers
  - B : the number of buffers (system capacity) (default-infinite)
  - K : the population size (default-infinite)

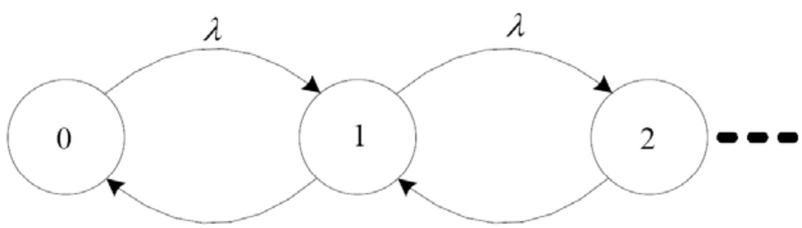
Notation (Kendall)

- Distributions are generally denoted by a one-letter symbol as follows:
  - M : Exponential
  - E_k : Erlang with parameter k
  - H_k : Hyper-exponential with parameter k
  - D : Deterministic
  - G : General

M/M/1 -Infinite Population, Single Server



M/M/1 State Diagram



Analytic Results

- Birth and death rates

  $\lambda(n) = \lambda, \;\; n \geq 0$

  $\mu(n) = \mu, \;\; n > 0$

- Solution

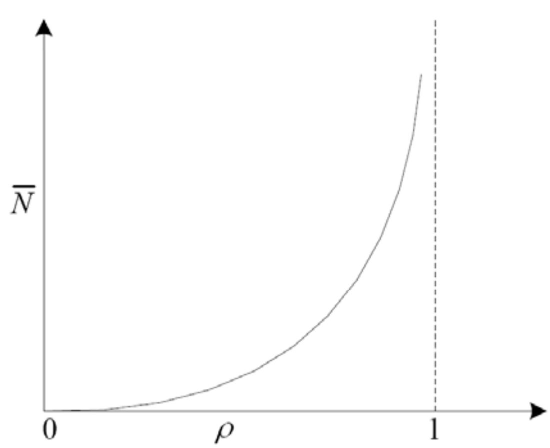
  $P(n) = P(0)\rho^n$ where $\rho = \lambda / \mu$ (traffic intensity)

  $G = \dfrac{1}{1-\rho}$ if $\rho < 1$

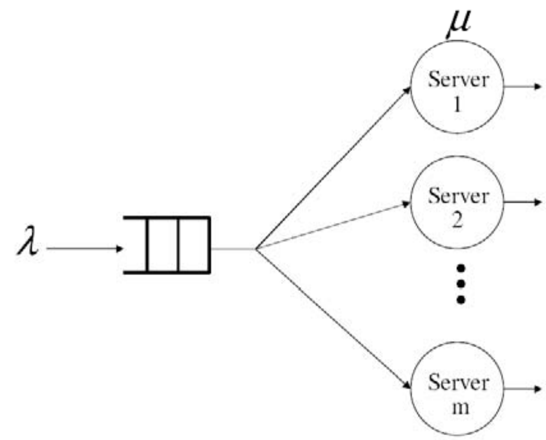
  Thus, $P(n) = (1-\rho)\rho^n, \rho < 1$

  $\Rightarrow \overline{N} = \dfrac{\rho}{1-\rho}$ and $\overline{T} = \dfrac{1}{\mu(1-\rho)}$

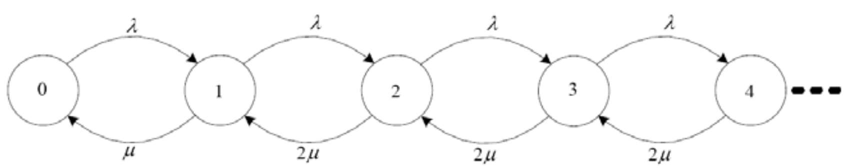
Traffic Intensity



M/M/m -Infinite Population, Multiple Servers



M/M/2 State Diagram



State Dependent Service Rate

- When j servers are busy,

$$\Pr(0 \text{ departure in } \Delta t) = [1 - \mu\Delta t + o(\Delta t)]^j$$
$$= 1 - j\mu\Delta t + o(\Delta t)$$
$$\Pr(1 \text{ departure in } \Delta t) = \binom{j}{1}[\mu\Delta t + o(\Delta t)][1 - \mu\Delta t + o(\Delta t)]^{j-1}$$
$$= j\mu\Delta t + o(\Delta t)$$
$$\Pr(2 \text{ or more departures in } \Delta t) = o(\Delta t)$$

Analytic Results

- Birth and death rates

$$\lambda(n) = \lambda, \quad n \geq 0$$
$$\mu(n) = \min(m, n)\mu, \quad n > 0$$

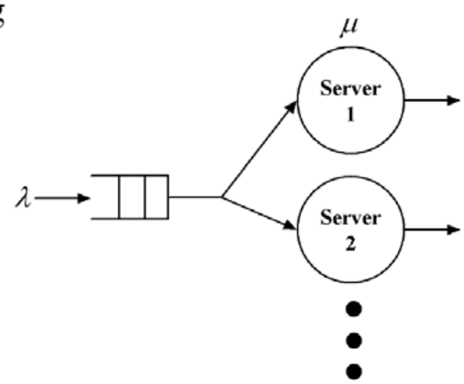
- Solution

$$P(n) = P(0)\prod_{i=0}^{n-1} \frac{\lambda(i)}{\mu(i+1)} = \begin{cases} P(0)\rho^n \dfrac{1}{n!} & n \leq m, \\ P(0)\rho^n \dfrac{1}{m!m^{n-m}} & n > m. \end{cases}$$

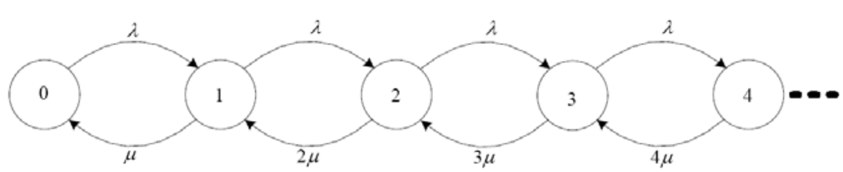
- Exercise: derive G and $\overline{T}$

M/M/∞-Infinite Population, Infinite Servers

- Potentially infinite number of servers - no queueing



M/M/∞State Diagram



Analytic Results

- Birth and death rates

  $\lambda(n) = \lambda, \quad n \geq 0$

  $\mu(n) = n\mu, \quad n > 0$

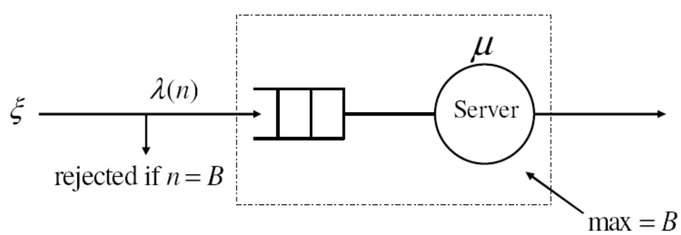
- Solution

  $P(n) = P(0)\rho^n \dfrac{1}{n!}$

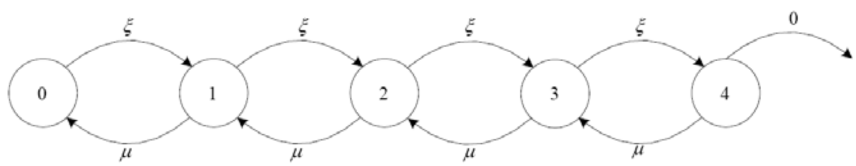
  $G = \sum_{n=0}^{\infty} \rho^n \dfrac{1}{n!} = e^{\rho}$

  Thus, $P(n) = \dfrac{\rho^n e^{-\rho}}{n!} \qquad \Rightarrow \bar{N} = \rho$ and $\bar{T} = \dfrac{1}{\mu}$

M/M/1/K –Infinite Population, Single Server with Finite Waiting Room



M/M/1/4 State Diagram



Analytic Results

- Birth and death rates

$$\lambda(n) = \begin{cases} \xi & 0 \le n < K, \\ 0 & n \ge K. \end{cases}$$

$$\mu(n) = \mu, \quad n > 0$$

- Solution

$$P(n) = \frac{1}{G}\rho^n \quad \text{where} \quad \rho = \frac{\xi}{\mu}$$

$$G = \sum_{n=0}^{K} \rho^n = \begin{cases} \dfrac{1-\rho^{K+1}}{1-\rho} & \rho \ne 1, \\ K+1 & \rho = 1. \end{cases}$$

Analytic Results

- Performance measures (for $\rho \neq 1$)

i. $\lambda = \sum_{n=0}^{\infty} \lambda(n)P(n) = \sum_{n=0}^{K-1} \xi P(n) = \xi \frac{1-\rho^K}{1-\rho^{K+1}}$

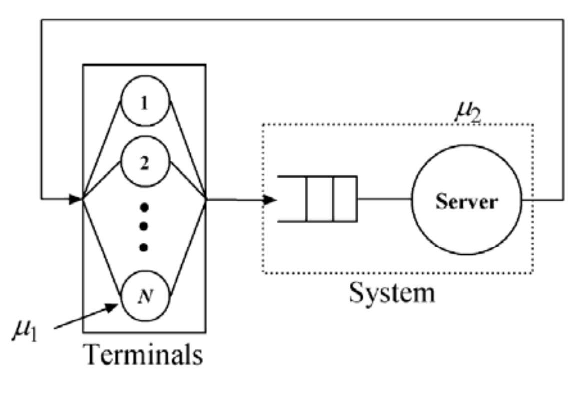
ii. $\overline{N} = \sum_{n=0}^{K} nP(n) = P(0)\sum_{n=0}^{K} n\rho^n = \frac{\rho[1-(K+1)\rho^K + K\rho^{K+1}]}{(1-\rho)(1-\rho^{K+1})}$

iii. $\overline{T}$ (of customers that are not rejected)

$$= \frac{\overline{N}}{\lambda} = \frac{[1-(K+1)\rho^K + K\rho^{K+1}]}{\mu(1-\rho)(1-\rho^K)}$$

Limiting behaviour: as $\xi \to \infty$, $\lambda \to \mu, \overline{N} \to K, \overline{T} \to \frac{K}{\mu}$

M/M/1/∞/N –Finite Population, Single Server



Analytic Results

- *Terminals*: service facility with no queueing
- When no. in system = n, no. in thinking state = N - n
- Birth and death rates

  $\lambda(n) = (N-n)\mu_1, \quad N > n \geq 0$

  $\mu(n) = \mu_2, \quad N \geq n \geq 1$
- Solution

  $$P(n) = P(0)\frac{N!}{(N-n)!}\left(\frac{\mu_1}{\mu_2}\right)^n$$

  $$G = \sum_{n=0}^{N}\frac{N!}{(N-n)!}\left(\frac{\mu_1}{\mu_2}\right)^n$$

Computation of Mean Response Time

- Standard approach

  $\overline{N} = \sum_{n=0}^{N} nP(n)$

  $\lambda = \sum_{n=0}^{N} \lambda(n)P(n)$

  Little's formula: $\overline{T} = \dfrac{\overline{N}}{\lambda}$

Computation of Mean Response Time

- Faster approach
  - from server utilization

    $\lambda = [1 - P(0)]\mu_2$

  - Little's formula at terminals

    $$N - \bar{N} = \frac{\lambda}{\mu_1} \quad \Rightarrow \bar{N} = N - \frac{\lambda}{\mu_1}$$

    $$\bar{T} = \frac{\bar{N}}{\lambda} = \frac{N}{\lambda} - \frac{1}{\mu_1} \quad \Rightarrow \bar{T} = \frac{N}{\mu_2[1 - P(0)]} - \frac{1}{\mu_1}$$