Analytic Modeling

Birth-Death Model

A Review -Random Variables

- A variable representing the outcome of a random activity, e.g., rolling a die
- Defines a mapping from outcome to value
- Has a range of values over which it can vary and a probability distribution with which it takes on these values
- Discrete random variable -can take on a finite or countable set of values, e.g., number of customers in system
- Continuous random variable -can take on values over a continuous interval, e.g., waiting time

Discrete Random Variables

- Discrete probability function, $P(n)$

  $P(n) = \Pr[\tilde{x} = n]$

  where $0 \le P(n) \le 1$ and $\sum_n P(n) = 1$

- Cumulative distribution function, $F(x)$

  $F(x) = \Pr[\tilde{x} \le x] = \sum_{n \le x} P(n)$

  where $F(0) = 0$

  $F(\infty) = 1$

  $F(b) \ge F(a)$ if $b \ge a$

  $F(b) - F(a) = \Pr[a < \tilde{x} < b] = \sum_{a<n<b} P(n)$

Discrete Random Variables

- Mean and variance

  mean: $\bar{x} = \sum_n nP(n)$

  variance: $\sigma^2 = \sum_n (n - \bar{x})^2 P(n)$

Continuous Random Variables

- Probability density function, $f(x)$

  $f(x) \geq 0$

  $\Pr[a \leq \tilde{x} \leq b] = \int_a^b f(x)dx$

  $\int_0^\infty f(x)dx = 1$

- Cumulative distribution function, $F(x)$

  $F(x) = \Pr[\tilde{x} \leq x] = \int_0^x f(y)dy$

  $f(x) = \dfrac{dF(x)}{dx}$

Continuous Random Variables

- Mean and variance

  mean: $\bar{x} = \int_0^\infty x f(x)dx$

  variance: $\sigma^2 = \int_0^\infty (x - \bar{x})^2 f(x)dx$

Exponential Distribution (with Parameter λ)

- $f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$

  $F(x) = 1 - e^{-\lambda x}$

  $\bar{x} = \frac{1}{\lambda}, \quad \sigma^2 = \frac{1}{\lambda^2}$

- Has memoryless property
- Coefficient of variation $= \frac{\sigma}{\bar{x}} = 1$

Memoryless Property

- $\Pr[x_0 < \tilde{x} \leq x + x_0 \mid \tilde{x} > x_0]$

$$= \frac{\Pr[x_0 < \tilde{x} < x + x_0]}{\Pr[\tilde{x} > x_0]}$$

$$= \frac{F(x + x_0) - F(x_0)}{1 - F(x_0)}$$

$$= \frac{1 - e^{-\lambda(x + x_0)} - 1 + e^{-\lambda x_0}}{1 - 1 + e^{-\lambda x_0}}$$

$$= 1 - e^{-\lambda x}$$

Memoryless Property -Interpretation

- Suppose interarrival time has exponential distribution with parameter λ
- Suppose an arrival occurs at time 0
- Given that there is no arrival between 0 and x_0, the time until the next arrival, measured from x_0, has exponential distribution with parameter λ
- Since time 0 and x_0 are arbitrarily chosen, the time until the next arrival, measured from any time instant, has exponential distribution with parameter λ
- *Note*: $1/\lambda$ is the mean interarrival time, and λ is the arrival rate

More on Exponential Distribution

- Suppose interarrival time has exponential distribution with parameter λ

$$\begin{aligned}
&\Pr[\text{no arrival in } (0,h)] \\
&= 1 - F(h) \\
&= e^{-\lambda h} \\
&= 1 - \lambda h + o(h)
\end{aligned}$$

where $o(h)$ has the property $\lim_{h \to 0} \dfrac{o(h)}{h} = 0$

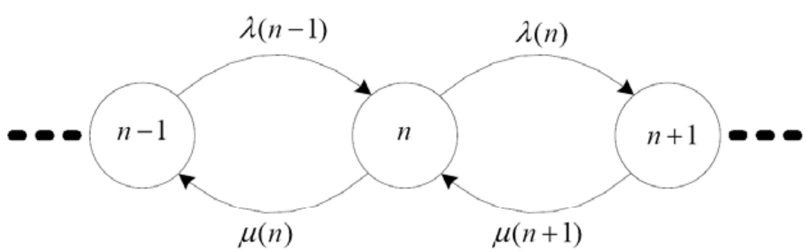
(an order of magnitude smaller than h)

More on Exponential Distribution

$\Pr[1 \text{ arrival in } (0,h)]$
$= \int_0^h \lambda e^{-\lambda y} dy \cdot e^{-\lambda(h-y)}$
$= \lambda h e^{-\lambda h}$
$= \lambda h + o(h)$

$\Pr[2 \text{ or more arrivals in } (0,h)]$
$= 1 - \Pr[\text{no arrival in } (0,h)] - \Pr[1 \text{ arrival in } (0,h)]$
$= o(h)$

Birth-Death Model



Birth-Death Model

- Queuing system with a single service facility
- Interarrival time is exponentially distributed
- Service time is exponentially distributed
- Arrival rate and service rate may be state dependent, i.e., a function of the state of the system

State Dependent Arrival Rate

- While the state of the system is j, interarrival time has exponential distribution with parameter $\lambda(j)$, e.g., $\lambda(j) = (N - j)\lambda$
  - $\lambda(j)$ is also the arrival rate
  - if state is changed, say to k, the arrival rate (or the parameter of exponential distribution) is immediately changed to $\lambda(k)$
- Useful for modeling the following features
  - finite population
  - finite queueing space

State Dependent Service Rate

- While the state of the system is j, service time has exponential distribution with parameter $\mu(j)$, e.g., $\mu(j) = j\mu$
  - $\mu(j)$ is also the service rate
  - if state is changed, say to k, the service rate (or the parameter of exponential distribution) is immediately changed to $\mu(k)$
- Useful for modeling the following features
  - multiple servers
  - infinite servers

Definition of Birth-Death Process

- Stochastic process $\tilde{n}(t)$ - a series of random variables indexed by time t
- Let $\tilde{n}(t)$ be the no. of customers in system at time t, and
- $q_{j,k}(h) = \Pr[\tilde{n}(t+h) = k \mid \tilde{n}(t) = j]$, which is called transition probability
- Definition $\tilde{n}(t)$ is a birth-death process if

  i. $q_{j,j+1}(h) = \lambda(j)h + o(h)$

  ii. $q_{j,j-1}(h) = \mu(j)h + o(h)$

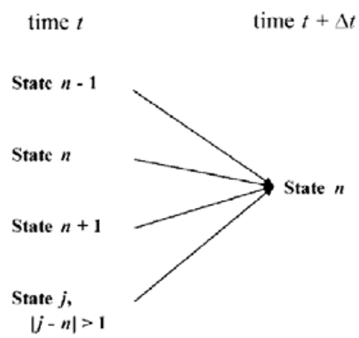
  iii. $q_{j,j}(h) = 1 - \lambda(j)h - \mu(j)h + o(h)$

  iv. $q_{j,k}(h) = o(h)$ for $|j-k| > 1$

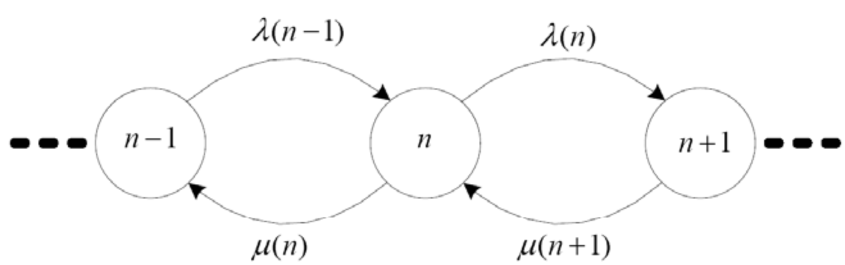
State Transition

- Let $P(n,t) = \Pr[\tilde{n}(t) = n]$

Consider $n > 1$



State Transitions



Analysis

$$P(n,t+\Delta t) = P(n-1,t)[\lambda(n-1)\Delta t + o(\Delta t)]$$
$$+ P(n,t)[1-\lambda(n)\Delta t - \mu(n)\Delta t + o(\Delta t)]$$
$$+ P(n+1,t)[\mu(n+1)\Delta t + o(\Delta t)] + o(\Delta t)$$

Rearranging and dividing by Δt,

$$\frac{P(n,t+\Delta t)-P(n,t)}{\Delta t} = \lambda(n-1)P(n-1,t) - [\lambda(n)+\mu(n)]P(n,t)$$
$$+ \mu(n+1)P(n+1,t) + \frac{o(\Delta t)}{\Delta t}$$

As $\Delta t \to 0$, we get, for $n > 0$

$$\frac{\partial P(n,t)}{\partial t} = \lambda(n-1)P(n-1,t) - [\lambda(n)+\mu(n)]P(n,t)$$
$$+ \mu(n+1)P(n+1,t)$$

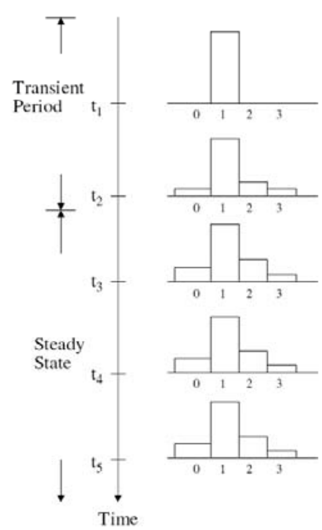
For $n = 1$

$$\frac{\partial P(0,t)}{\partial t} = -\lambda(0)P(0,t) + \mu(1)P(1,t)$$

Remarks

- Analysis leads to a set of differential-difference equations which are difficult to solve in general
- Normal practice is to obtain steady state results for $P(n, t)$ (i.e., $P(n, t)$ as $t \rightarrow \infty$)
  - steady state results are much easier to obtain
  - useful if one is interested in the long-term behavior of the system
  - usually considered as adequate when high-level performance estimates are required

Steady State Behavior



Steady State Probability

- Define steady state probability distribution

$$P(n) = \lim_{t\to\infty} P(n,t) \qquad \Rightarrow \frac{\partial P(n,t)}{\partial t} = 0$$

- $P(n)$ is the long-run probability that there are n customers in the system
- We have a set of *balance equations* (at steady state):

For $n > 0$,

$$[\lambda(n) + \mu(n)]P(n) = \lambda(n-1)P(n-1) + \mu(n+1)P(n+1)$$

For $n = 0$,

$$\lambda(0)P(0) = \mu(1)P(1)$$

- Solving the above equations

$$P(n+1) = \frac{\lambda(n)}{\mu(n+1)}P(n), \quad n \geq 0$$

$$\Rightarrow P(n) = P(0)\prod_{i=0}^{n-1}\frac{\lambda(i)}{\mu(i+1)}, \quad n \geq 0$$

Since $\sum_n P(n) = 1$, then

$$P(0)\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}\frac{\lambda(i)}{\mu(i+1)} = 1$$

Normalization Constant, G

- Let $G = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda(i)}{\mu(i+1)}$
- If the series G converges, $P(0) = 1/G$, steady state solution exists and
  $P(n) = P(0) \prod_{i=0}^{n-1} \frac{\lambda(i)}{\mu(i+1)}, \quad n \geq 0$
- Otherwise, $P(n) = 0$, for $n = 0, 1, \ldots$