Assignment 1, Probability and Stochastic Process

Due Date is: Sept. 14th 2021

Question 1, About Exponential Distribution

Exponential distribution is a very important distribution in this course. We will use it frequently in our future lectures. Suppose a non-negative real valued random variable X obeys an exponential distribution with parameter μ. That is, the probability density function of X is $f(X = x) = \mu e^{-\mu x}$, $x \geq 0$.

a) Prove that X has the memoryless property. That is, the p.d.f. $f(X = x + t | X > t)$, $x > x_0$, also has the same form as $f(X = x)$.

b) Calculate the coefficient of variability of X, $C^2\{X\}$, where $C^2\{X\} := \frac{Var\{X\}}{(E\{X\})^2}$. (write the detailed calculation process) Note: please use Laplace transform.

c) For the two independently exponentially distributed random variables X_1 and X_2 with parameter μ_1 and μ_2, respectively, calculate the probability $P(X_1 < X_2)$.

d) Suppose three exponential distributed random variable X_1, X_2, X_3 with parameter μ_1, μ_2, μ_3, respectively. They are independent. Determine the distribution of random variable $Y = \min\{X_1, X_2, X_3\}$.

e) Similar to the conditions in the previous sub-question, determine the distribution of random variable $Z = \max\{X_1, X_2, X_3\}$.

Question 2, About Poisson Distribution

Poisson distribution is another important distribution. It has a closed relation with exponential distribution. Suppose X is a non-negative integer valued random variable and obeys Poisson distribution with parameter λ. That is, $Pr\{X = n\} = \frac{\lambda^n e^{-\lambda}}{n!}$, $n = 0, 1, \ldots$. Answer the following questions.

a) Calculate the mean and variance of X. Note: please use the Z transform.

b) Suppose X_1 and X_2 are two Poisson distributed random variables with parameter λ_1 and λ_2, respectively. Determine the distribution of integer valued random variable $Y := X_1 + X_2$.

Question 3, About elementary queue

Consider a service facility with single server and infinite buffer. The customer arrival is a Poisson process with rate λ. The service time of each customer is a series of i.i.d. (independently and identically distributed) random numbers, denoted as X.

For these 2 cases: i. $X = c$ which is a constant; ii. X is exponentially distributed with mean $1/\mu$, please calculate respectively:

1). the probability P that the second arriving customer will not have to wait.

2). the average waiting time of the second arriving customer, W

Question 4, About Markov Chain

Consider a homogeneous DTMC whose state transition diagram is as follows.

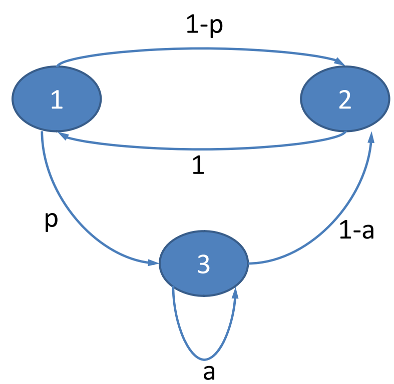

Figure 1: The state transition diagram of a Markov chain.

a) Write the transition probability matrix P of this DTMC.

b) Under what conditions will the chain be irreducible?
Under what conditions will the chain be aperiodic?

c) Calculate the steady state probability of system states.

d) Calculate the mean recurrence time of state 2.

e) For which values of a and p, we have $\pi_1 = \pi_2 = \pi_3$?