**[Question 1]**

A supermarket is going to have a new store design and remodel the store as follows. The checkout-counter is different from the usual case: the store has been remodeled to include a checkout "lounge". As customers complete their shopping, they enter the lounge with their carts and, if all checkers are busy, receive a number. They park their carts and take a seat. When a checker is free, the next number is called and the customer with that number enters the available checkout counter. The store has been enlarged so that for practical purposes, there is no limit on either the number of shoppers that can be in the food aisles or the number that can wait in the lounge, even during the peak periods.

The manager estimates that during the peak hours customers arrive according to a Poisson process at a mean rate of 60/hour and it takes a customer, on average, 45 minutes to fill a shopping cart, the filling times being approximately exponentially distributed. Furthermore, the checkout times are also approximately exponentially distributed with a mean of 4 minutes.

Hint, for M/M/c queue, the probability that all servers are idle is

$$p=\left[\frac{r^c}{c!(1-\rho)}+\sum_{n=0}^{c-1}\frac{r^n}{n!}\right]^{-1},\quad r=\lambda/\mu,\ \rho=r/c;$$

the customer average response time is $W=\frac{1}{\mu}+\left[\frac{r^c}{c!c\mu(1-\rho)^2}\right]p$.

Question:

**1.** How many customers will be in the shopping area on average? **(1 point)**

[Answer] for the shopping area, this is an M/M/∞ queue. The average number of customers is L1=60/hour*45min=45.

2. What's the minimum number of checkout counters required? **(1 point)**

[Answer] for the checkout period including the waiting lounge, it is an M/M/c queue with arrival rate 60/hour. The minimum number of checkout counters is: c/4min>60/hour → c>4.

3. Suppose the current number of the counters is 5:

a) What is the average waiting time in the lounge? **(1 point)**

[Answer] for the M/M/5 queue, we have

$$P_0=\left[\frac{\left(60\cdot\frac{4}{60}\right)^5}{5!\left(1-60\cdot\frac{4}{60}\cdot\frac{1}{5}\right)}+\sum_{n=0}^{4}\frac{\left(60\cdot\frac{4}{60}\right)^n}{n!}\right]^{-1}=\left[\frac{4^5}{5!0.2}+\sum_{n=0}^{4}\frac{4^n}{n!}\right]^{-1}=0.0130.$$

$$W_q=\left[\frac{\left(60\cdot\frac{4}{60}\right)^5}{5!5\frac{60}{4}\left(1-60\cdot\frac{4}{60}\cdot\frac{1}{5}\right)^2}\right]P_0=2.2165\text{ min.}$$

b) How many customers will be in the lounge? **(1 point)**

[Answer] queue length of this model is $L_q=\lambda W_q=60*2.2165/60=2.2165$.

c) How many customers will be in the entire supermarket? **(1 point)**

[Answer] the number of customers at checkout counters is L2=60/hour*4min=4.
So, the number of total customers in the entire supermarket is L=L1+L2+Lq=51.2165.

**[Question 2]**

There are 3 different queueing systems with the same customer requests and service abilities, compare their performance metrics. Assumption: the customer arrivals for this 3 systems are the same as Poisson process with rate $\lambda$=15. The total service resources *R* for this 3 systems are also identical, denoting as exponential service time with mean $1/\mu$=0.05.

*System1:* the system has 3 identical and independent servers which share the common total service resource *R*. The customer arrivals will be evenly assigned to these 3 servers, that is, 3 servers have their own queues.

*System2:* the system has 3 identical servers which share the common total service resource *R*. These 3 servers only have one common queue. Whenever a waiting customer in the queue finds a server idle, it will instantly enter that server to get service.

*System3:* the system only has one server and one queue.

Question:

1. Choose proper queueing models for these 3 different systems and calculate their customer average response time, respectively. **(3 points)**

   [Answer] the first one is 3 identical M/M/1 queues, each one with arrival rate $\lambda/3$ and service rate $\mu/3$. So, the average response time is $W_1=1/(\mu/3-\lambda/3)=0.6$.

   The 2nd system is an M/M/3 queue with arrival rate $\lambda$ and service rate $\mu/3$. So, the average response time is

   $$W_2 = \frac{3}{\mu} + \left[\frac{\left(\frac{3\lambda}{\mu}\right)^3}{3!\mu\left(1-\frac{\lambda}{\mu}\right)^2}\right]\left[\frac{\left(\frac{3\lambda}{\mu}\right)^3}{3!\left(1-\frac{\lambda}{\mu}\right)} + \sum_{n=0}^{2}\frac{\left(\frac{3\lambda}{\mu}\right)^n}{n!}\right]^{-1} = 3/20 + \left[\frac{\left(\frac{9}{4}\right)^3}{3!20\left(1-\frac{3}{4}\right)^2}\right]\left[\frac{\left(\frac{9}{4}\right)^3}{3!\left(1-\frac{3}{4}\right)} + \sum_{n=0}^{2}\frac{\left(\frac{9}{4}\right)^n}{n!}\right]^{-1} = 0.2636$$

   The 3rd system is an M/M/1 queue with arrival rate $\lambda$ and service rate $\mu$. So, the average response time is $W_3=1/(\mu-\lambda)=0.2$.

2. Compare the response times of these 3 systems and discuss the possible rule on how to design a queueing system (how to allocate the budgeted service resource and how to design the waiting room and waiting rule). **(2 points)**

   [Answer]

   $W_3<W_1$, $W_2<W_1$, it means we should keep one common waiting line if possible;

   $W_3<W_2$, it means we should integrate all the service resources together to one server if possible.